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RESIDUALLY LINEAR GROUPS

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Abstract. In this work, we investigate the class of residually linear groups and survey the literature for considerable group classes that are included in. Various properties and characterizations of this class are provided and the relationship with residual finiteness is studied.

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1. Introduction

Recall that a group is called linear over a field $\Bbbk$ provided that, it is isomorphic to a subgroup of the general linear group $\mathrm{GL_n}(\Bbbk)$ for some $n \in \mathbb{N}$.
The Peter-Weyl Theorem [17] concerns a compact Lie group G, one of its components asserts that:

$$(1) \qquad \forall \;\; e \neq g \in G \; , \; \exists \, \sigma \in \mathrm{rep}(G) \;\; | \;\; \sigma(g) \neq 1.$$

One of the direct results of this theorem is linearity of G: in fact, claim (1) implies that *any compact Lie group admits a finite dimensional faithful representation* [4, Corollary 4.4,

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page 34]. The abstraction of the above condition (1) yields the property of residual linearity. As not all groups can be faithfully represented by matrices acting on finite dimensional vector spaces, and hence unveiling essential features and symmetries hidden within the group's structure; residual linearity addresses this limitation by approximating an arbitrary group by linear groups. This property provides connections between the abstract nature of group theory and the concrete realm of linear algebra.

Residual properties play prominent roles in discrete group theory in understanding groups' intrinsic properties, such as the theorems of Magnus [11] and Malcev [12] affirming residual nilpotence of free groups and generalizing this result to free products, and the work [9] of Gruenberg in 1955, on residual properties of infinite soluble groups. While many residual properties have been intensively studied, e.g., residual finiteness [2, 8, 11, 14], residual smallness [16]; a probably less attention is devoted to explore the concept of residual linearity, where we find only few references treating this topic on itself, this shows up mainly in the work [13] of Menal in 1978, in order to determine when a residually linear group is residually finite. He proved that residual linearity of a nilpotent group G along with the condition of having finitely generated center and commutator subgroups, implies residual finiteness of G.

By the very mentioned relationship of residual linearity and Peter-Weyl Theorem, this study sheds light on a potential application to the Tannaka-Krein duality for abstract groups especially via the Hopf algebra of representative functions approach, see [4] for a nice exposition.

Throughout the sequel, if not mentioned otherwise, the base field, denoted by $\Bbbk$, is a fixed arbitrary one. $\mathrm{rep}(G)$

will always denote the category of finite dimensional representations of G over $\mathbb{k}$, namely the category whose objects are homomorphisms $\rho : G \longrightarrow \mathrm{GL}(V)$ of groups, from G to the group of automorphisms of a finite dimensional $\mathbb{k}-$vector space V (we write $\mathrm{GL}_{\mathrm{n}}(V)$ to indicate the dimension n of V).

2. Residual linearity

In what follows, we study the structure, stability properties: under taking subgroups, direct products, projective limits, etc, and some characterizations of the class of residually linear groups. We investigate the relationship with residual finiteness and illustrate the situation with examples from the literature, that show what some considerable group classes are included in this class and what are not.

Definition 2.1. *Let G be an abstract group with neutral element e. G is called residually linear (over $\mathbb{k}$), provided that for every $e \neq g \in G$, there exists a finite dimensional representation $\rho \in \mathrm{rep}(G)$ of G over $\mathbb{k}$, such that $\rho(g) \neq 1$.*

Note that this notion depends on the base field, some groups are residually linear over any arbitrary field and some others are only over specific ones, or over no one. For example, $(\mathbb{Q}, +)$ is residually linear over $\mathbb{R}$, but not over finite fields, as it does not have any proper subgroup of finite index. In general, divisible groups can not be residually linear over any finite field, in light of Remark 2.21.

Example 2.2. *The underlying multiplicative group $\mathbb{k}^*$ of $\mathbb{k}$ is residually linear, since $\mathbb{k}^* \simeq \mathrm{GL}_1(\mathbb{k})$. In particular, $\mathbb{Q}^*, \mathbb{R}^*$ and $\mathbb{C}^*$ are residually linear, over $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively.*

Proposition 2.3. *Every linear group is residually linear (over the same field).*

Proof. Immediate as any linear group admits a finite dimensional faithful representation. □

Theorem 2.4. *Let G be a group. The following statements are equivalent.*

(*a*) *G is residually linear.*
(*b*) *$I_G = \{e\}$.*
(*c*) *For any elements $g, h \in G$, such that $g \neq h$, there exists a finite dimensional representation $\rho \in \mathrm{rep}(G)$, such that $\rho(g) \neq \rho(h)$.*
(*d*) *For every $e \neq g \in G$, there exists a normal subgroup $N_g \lhd G$, such that $g \notin N_g$ and G/N_g is linear.*
(*e*) *$\bigcap\limits_{N \in \mathcal{N}} N = \{e\}$, where $\mathcal{N} = \{N \lhd G,\ G/N \text{ is linear}\}$.*

To prove the above theorem, we will need the following two lemmas.

Lemma 2.5. *Let G be an abstract group and set $I_G := \bigcap\limits_{\rho \in \mathrm{rep}(G)} \ker(\rho)$. Then*

(*a*) *I_G is a normal subgroup of G.*
(*b*) *G/I_G is residually linear.*

Proof. (*a*) Straightforward.
(*b*) Let $\overline{g} \in G/I_G$, non neutral. Then $g \in G \setminus I_G$. By definition of I_G, there exists $\rho_g \in \mathrm{rep}(G)$, such that $\rho_g(g) \neq 1$. Since $I_G \subseteq \ker(\rho_g)$, there exists, by the universal property of quotient groups, a morphism ρ'_g of groups, such that $\rho'_g(\overline{g}) = \rho_g(g)$. This finishes the proof. □

Lemma 2.6. *Let G be a group and $N \lhd G$. If G/N is residually linear, then $I_G \subseteq N$.*

Proof. That is trivial from the definition of I_G in the above Lemma 2.5. □

Now, we prove Theorem 2.4.

Proof. $(a) \Leftrightarrow (b)$ This holds by the universal property of quotient groups, using Lemma 2.5.
$(a) \Leftrightarrow (c)$ Observe that $g \neq h \Leftrightarrow gh^{-1} \neq e$, then apply the previous equivalence.
$(d) \Rightarrow (a)$ Let $e \neq g \in G$, pick any finite dimensional faithful representation of G/N_g and compose it with the canonical projection.
$(a) \Rightarrow (d)$ For every $e \neq g \in G$, consider the normal subgroup $N_g = \ker(\rho_g)$, for some $\rho_g \in \mathrm{rep}(G)$ with $\rho_g(g) \neq 1$, then the result holds by the universal property of quotient groups related to ρ_g.
$(d) \Rightarrow (e)$ Let $e \neq g \in \bigcap_{N \in \mathcal{N}} N$, there exists $N_g \lhd G$ with $g \notin N_g$ and G/N_g is linear. But $N_g \in \mathcal{N}$, namely $g \in N_g$, which is a contradiction. Therefore, (e) holds.
$(e) \Rightarrow (d)$ Assume that there is $e \neq g \in G$, such that for any $H \lhd G$ with $g \notin H$ and G/H is not linear. So, $g \in N$ for any $N \lhd G$, such that G/N is linear, but such latter normal subgroups have trivial intersection by hypothesis, then $g = e$, hence a contradiction. Thus, (d) holds. □

Corollary 2.7. *Let G be a group such that for every $e \neq g \in G$, there exists $N_g \lhd G$, such that $g \notin N_g$. If G/H is residually linear for any proper normal subgroup H of G, then G is residually linear.*

Proof. Assume G is not residually linear. Let $e \neq g \in I_G$, there exists $N_g \not\ni g$, such that $N_g \lhd G$. But G/N_g being residually linear by assumption implies, by Lemma 2.6, that $I_G \subseteq N_g$, which is a contradiction. Thus, G is residually linear. □

Theorem 2.8. *Let $(G_i)_{i \in I}$ be a family of groups. The following statements are equivalent.*

(a) *G_i is residually linear for every $i \in I$.*

(*b*) *The direct product* $\prod_{i\in I} G_i$ *is a residually linear group.*

Proof. $(a) \Rightarrow (b)$ Let $x = (x_i) \in G = \prod_{i\in I} G_i$ be a non neutral element of G, there exists $e_{G_{i_0}} \neq x_{i_0} \in G_{i_0}$ for some $i_0 \in I$. But G_{i_0} is residually linear, hence there exists a representation $\rho_{x_{i_0}} \in \mathrm{rep}(G_{i_0})$ on a finite dimensional $\Bbbk-$vector space V, such that $\rho_{x_{i_0}}(x_{i_0}) \neq 1$. Consider the morphism $\rho_{x_{i_0}}\pi : G \longrightarrow \mathrm{GL_n}(V)$ of groups, where π is the canonical projection on G_{i_0}. Then, $\rho_{x_{i_0}}\pi$ is a finite dimensional representation of G, and $\rho_{x_{i_0}}\pi(x) \neq 1$.
$(b) \Rightarrow (a)$ This holds since every representation of $\prod_{i\in I} G_i$, composed with the canonical injection of G_i in $\prod_{i\in I} G_i$, is a representation of G_i. □

Corollary 2.9. *Let* $(G_i)_{i\in I}$ *be a family of groups. Then,* $I_{\prod_{i\in I} G_i} = \prod_{i\in I} I_{G_i}$.

Proof. Let $\pi_i : G_i \longrightarrow G_i/I_{G_i}$ be the canonical projection, for every $i \in I$, and consider $\pi = \prod_{i\in I} \pi_i$. Then, π is an epimorphism of groups, and $\ker(\pi) = \prod_{i\in I} \ker(\pi_i) = \prod_{i\in I} I_{G_i}$. So $\prod_{i\in I} G_i/\prod_{i\in I} I_{G_i} \simeq \prod_{i\in I} (G_i/I_{G_i})$. By Lemma 2.5 (*b*) and Theorem 2.8, $\prod_{i\in I} G_i/\prod_{i\in I} I_{G_i}$ is residually linear. The Lemma 2.6 implies that $I_{\prod_{i\in I} G_i} \subseteq \prod_{i\in I} I_{G_i}$.
Let us now prove the other inclusion $\prod_{i\in I} I_{G_i} \subseteq I_{\prod_{i\in I} G_i}$. Let $x \in \prod_{i\in I} I_{G_i}$, then $x = (x_i)$ with $x_i \in I_{G_i}$ for every $i \in I$. We can write x as $x = \prod_{j\in I} (x_{ij})$ (component-wise product),

where $x_{ij} = x_i$ if $j = i$ and $x_{ij} = e_{G_i}$ otherwise (e_{G_i} designates the neutral element of G_i). Since every representation of $\prod_{i\in I} G_i$, composed with the canonical injection ι, is a representation of G_i, for every $i \in I$; for every $\rho \in \text{rep}\Big(\prod_{i\in I} G_i\Big)$, we get $\rho(x) = \rho\Big(\prod_{j\in I}(x_{ij})\Big) = \prod_{j\in I}\Big(\rho\ \iota(x_i)\Big) = 1$, so $\prod_{i\in I} I_{G_i} \subseteq I_{\prod_{i\in I} G_i}$, and this completes the proof. □

Remark 2.10. *The obtainment of a residually linear group from a non residually linear group as mentioned in Lemma 2.5 (b), which occurred by "deleting" the non residually linear part, can be stated more generally as follows. Let G and H be two groups such that G is residually linear and H not. Then, $G \times H$ is not residually linear by Theorem 2.8, but $(G \times H)/H'$ is so, where $H' = \{(e_G, h), h \in H\}$. Indeed, $H' \lhd (G \times H)$ and $(G \times H)/H' \simeq G$.*

Corollary 2.11. *Let $(G_i)_{i\in I}$ be a family of groups. Then, the direct sum $\bigoplus_{i\in I} G_i$ is a residually linear group if and only if G_i is residually linear for every $i \in I$.*

Proof. The direct sum is a subgroup of the direct product, hence the result holds by Theorem 2.8.

Conversely, this is immediate as every G_i is a subgroup of $\bigoplus_{i\in I} G_i$. □

Corollary 2.12. *The projective limit of a projective system of residually linear groups is a residually linear group.*

Proof. Let $(G_i)_{i\in I}$ be a projective system of residually linear groups. By construction, the projective limit group $G = \varprojlim G_i$ is a subgroup of $\prod_{i\in I} G_i$. Hence, the result holds by Theorem 2.8. □

Corollary 2.13. *Let G be a group, $N_1 \lhd G$ and $N_2 \lhd G$, such that G/N_1 and G/N_2 are residually linear. Then, $G/(N_1 \cap N_2)$ is residually linear.*

Proof. Note that $G/(N_1 \cap N_2)$ can be identified with a subgroup of $G/N_1 \times G/N_2$, then use Theorem 2.8. □

Corollary 2.14. *Let $\alpha : G \hookrightarrow Q$ and $\beta : H \hookrightarrow Q$ be two monomorphisms of groups, such that Q is residually linear. Then, the fiber product $G \times_Q H$ is residually linear.*

Proof. Immediate from Theorem 2.8. □

Theorem 2.15. *Let G be a group. The following statements are equivalent.*

(*a*) *G is residually linear.*
(*b*) *There exists a family of linear groups $(G_i)_{i \in I}$, such that G is a subdirect product of $\prod_{i \in I} G_i$.*

Proof. $(a) \Rightarrow (b)$ G residually linear implies that for every $e \neq x \in G$, there exists a representation ρ_x of G on a finite dimensional $\Bbbk-$vector space V_x, such that $\rho_x(x) \neq 1$. Let $\rho = \prod_{x \in G \setminus \{e\}} \rho_x$ and $\iota : G \longrightarrow \prod_{x \in G \setminus \{e\}} G$ be the canonical injection, then $\rho\ \iota : G \longrightarrow \prod_{x \in G \setminus \{e\}} \mathrm{GL}(V_x)$ is injective, hence G is isomorphic to a subgroup of the direct product $\prod_{x \in G \setminus \{e\}} \mathrm{GL}(V_x)$ of linear groups.
$(b) \Rightarrow (a)$ This holds by Proposition 2.3 and Theorem 2.8, since G is a subgroup of a residually linear group. □

We resume some considerable group classes that are included in the class of residually linear groups.

Example 2.16. *Braid groups are residually linear. In fact, they are linear* [10].

Proposition 2.17. *Every finite group is residually linear.*

Proof. Every finite group is in fact linear (over any field) by the Cayley's Theorem [1, Theorem 7.1.3, page 195] and the standard embedding of any symmetric group $\mathrm{S_n}$ into $\mathrm{GL_n}(\Bbbk)$. □

Recall that a group G is called residually finite provided that for every $e \neq g \in G$, there exists a morphism of groups: $\alpha_g : G \longrightarrow H$, such that H is finite and g does not belong to the kernel of α_g.

Proposition 2.18. *Every residually finite group is residually linear.*

Proof. In fact, for every $e_G \neq g \in G$, there exists a morphism $h_g : G \longrightarrow H$ of groups, with H finite, such that $h_g(g) \neq e_H$. But H is residually linear by Proposition 2.17, there exists $\rho \in \mathrm{rep}(H)$, such that $\rho(h_g(g)) \neq 1$. Hence, $\rho h_g \in \mathrm{rep}(G)$ and $\rho h_g(g) \neq 1$. □

Example 2.19. *Free, cyclic, polycyclic, profinite, supersolvable, finitely generated nilpotent and automata groups are all residually linear groups. In fact, these groups are residually finite* [11].

Proposition 2.20. *Let G be a finitely generated group. The following statements are equivalent.*

(*a*) *G is residually linear.*
(*b*) *G is residually finite.*

Proof. $(b) \Rightarrow (a)$ This is always true without assuming G finitely generated by Proposition 2.18.
$(a) \Rightarrow (b)$ For every $e \neq g \in G$, there exists $g \notin N_g \lhd G$ such that G/N_g is linear, by Theorem 2.4. But G is finitely generated, so G/N_g is a finitely generated linear group. By Malcev [12], any finitely generated linear group is residually

finite, and then residually linear. Hence, there exists a morphism $f_g \in \mathrm{rep}(G/N_g)$ of groups, such that $f_g(\overline{g}) \neq 1$. As required by considering the composite πf_g, where π is the canonical projection. □

Remark 2.21. *Let G be a group and $\Bbbk$ a finite field. The following statements are equivalent.*

- (a) *G is residually linear (over $\Bbbk$).*
- (b) *G is residually finite.*

It suffices to show that $(a) \Rightarrow (b)$, which is immediate as $\mathrm{GL_n}(\Bbbk)$ is finite for every $n \in \mathbb{N}^*$.
Assume moreover that any proper subgroup of G is of infinite index. Then, the following statements are equivalent.

- (a) *G is residually linear.*
- (b) *G is linear.*
- (c) *G is residually finite.*
- (d) *G is finite.*

In particular, if G is infinite, then its Hopf algebra of representative functions $\mathcal{R}_{\Bbbk}(G)$ is trivial, namely $\mathcal{R}_{\Bbbk}(G) \simeq \Bbbk$.

Remark 2.22. *Let G be a finitely generated group. Then, G/I_G is residually finite by Proposition 2.20 and Lemma 2.5 (b).*

The group of automorphisms of a finitely generated residually finite group is residually finite [2]. The same result holds for residually linear groups.

Corollary 2.23. *The group of automorphisms of a finitely generated residually linear group is residually linear.*

Proof. Direct by combining Propositions 2.18 and 2.20. □

The next example shows that the properties "residually linear" and "linear" coincide for some classes of groups.

Example 2.24. *Let G be an infinite simple group. Then, the following statements are equivalent.*

(a) *G is residually linear.*
(b) *G is linear.*
(c) *G admits a non trivial finite dimensional representation.*

For example, algebraically closed groups, which are acyclic, infinitely generated and simple due to Neumann [15].

However, generally, the class of residually linear groups contains strictly that of linear groups, as shows the following example.

Example 2.25. *Let G be a free group of rank $n \geq 3$, then the group $\mathrm{Aut}(G)$ of automorphisms of G is residually linear, but not linear. See also Example 2.27 (b). In fact, G being free, it is residually finite, so $\mathrm{Aut}(G)$ is also residually finite since G is finitely generated* [2], *hence $\mathrm{Aut}(G)$ is residually linear by Proposition 2.18. However, $\mathrm{Aut}(G)$ is not linear by* [7].

Now we give some examples of non residually linear groups.

Example 2.26. *Every group with no nontrivial representations is a non residually linear group, see* [3] *for such groups, called counter-linear. In particular, binate groups and the automorphisms groups of de la Harp and McDuff* [5] *are acyclic non residually linear groups.*

The above groups are having trivial Hopf algebras of representative functions, namely isomorphic to $\mathbb{k}$.

For finitely generated and (co-)Hopfian groups, the following examples show that these classes are not comparable with that of residually linear groups.

Example 2.27.

(a) *The Higman's four generator, four relator group is an infinite, finitely generated non residually linear group, since it is counter-linear* [3].

(b) *The group* $\langle a, t \mid a^{t^2} = a^2 \rangle$ *presented in* [6] *as the first example of a non-linear residually finite* 1*-related group is a finitely generated residually linear group.*

(c) *Any finitely generated abelian group is residually linear. In fact, it is residually finite.*

Example 2.28.

(a) *By Malcev* [12], *every finitely generated residually finite group is Hopfian, also every finitely generated abelian group is known to be Hopfian. Thus, these are examples of residually linear Hopfian groups.*

(b) *Free groups of infinite rank are neither Hopfian, nor co-Hopfian, but these are residually linear.*

If we call a subgroup H of a group G, a residually linear subgroup of G provided that for any $e \neq h \in H$, there exists a finite dimensional representation ρ of G, such that $\rho(h) \neq 1$.

Proposition 2.29. *Let* G *be a group and* H *a subgroup of* G*. Then,* H *is a residually linear subgroup of* G *if and only if* $I_G \cap H = \{e\}$.

Proof. Straightforward. □

Remark 2.30. *Let* G *be a group and* H *a subgroup of* G.

(a) *If* G *is residually linear, then* H *is residually linear both as a group and as a subgroup.*

(b) *There is a bijective correspondence between the sets:*

$$\{\mathrm{rep}(H)\} \leftrightarrows \{\mathrm{rep}(I_G \times H)\}.$$

In fact, every finite dimensional representation of $I_G \times H$ *is a finite dimensional representation of* H*, seen as a subgroup by the identification* $H \longrightarrow I_G \times H$*,* $h \mapsto (e, h)$*.*

Conversely, every representation $\rho \in \operatorname{rep}(H)$*, can be extended to a representation* $\overline{\rho} \in \operatorname{rep}(I_G \times H)$ *as follows:* $\overline{\rho}\big((a,h)\big) := \rho(h)$*; that is, if* (a,h)*,* $(a',h') \in I_G \times H$*, then* $\overline{\rho}\big((a,h)(a',h')\big) = \overline{\rho}(aa', hh') = \rho(hh') = \rho(h)\rho(h') = \overline{\rho}(a,h)\overline{\rho}(a',h')$*.*

(c) *If* H *is residually linear as a subgroup of* G*, then it is residually linear as a group. In other words, we have the following inclusion*

$$I_H \subseteq I_G \cap H.$$

The equality holds for example, if G *is residually linear, or more generally as shows the next result.*

Proposition 2.31. *Let* H *be a subgroup of a non residually linear group* G*. Then,* H *is a residually linear group if and only if it is so as a subgroup of* $I_G \times H$*.*

Proof. We identify H with $\{e\} \times H$. Let $e \neq h \in H$, there exists $\rho \in \operatorname{rep}(H)$, such that $\rho(h) \neq 1$. By Remark 2.30 (b), ρ extends to $\overline{\rho} \in \operatorname{rep}(I_G \times H)$ and we have $\overline{\rho}\big((e,h)\big) = \rho(h) \neq 1$.
The converse implication holds by Remark 2.30 (c). Namely, we have $I_H = I_{(I_G \times H)} \cap H$. □

Remark 2.32. *Let* H *be a subgroup of a group* G *and* $g \in G$*. Then,* H *is a residually linear subgroup of* G *if and only if* gHg^{-1} *is a residually linear subgroup of* G*.*

Example 2.33. *Let* $\mathbb{k} = \mathbb{R}$*.*
(a) *The additive group* $(\mathbb{R}, +)$ *is residually linear. In fact,* $(\mathbb{R}, +)$ *is isomorphic to* $(\mathbb{R}^*_+, \times)$*, which is residually linear by Remark 2.30* (a)*, being a subgroup of* $(\mathbb{R}^*, \times)$*, treated in*

Example 2.2.
(b) *The additive groups* $(\mathbb{Z},+)$ *and* $(\mathbb{Q},+)$ *are residually linear, by Remark 2.30* (a)*, being subgroups of* $(\mathbb{R},+)$*.*
(c) *The additive group* $(\mathbb{C},+)$ *is residually linear. In fact,* $(\mathbb{C},+) \simeq (\mathbb{R},+) \times (\mathbb{R},+)$*, then use Theorem 2.8.*
Note that $(\mathbb{Q},+)$*,* $(\mathbb{R},+)$ *and* $(\mathbb{C},+)$ *are residually linear, but not residually finite as they are divisible. In contrast,* $(\mathbb{Z},+)$ *is residually finite, it suffices to consider, for any* $n_0 \in \mathbb{Z}^*$*, the quotient* $\mathbb{Z}/n\mathbb{Z}$ *for any integer* n *with* $n > |n_0|$*.*

Remark 2.34. *The class of residually linear groups is not stable under taking quotients. For example, picking any non residually linear group* G*, we know that* G *is isomorphic to a quotient of a free group* F *(see e.g.,* [1, Corollary 7.10.14 (i) and (ii), page 215]*), but* F *is residually linear.*

Proposition 2.35. *Let* G *be a residually linear group and* $N \triangleleft G$*, such that* $N \subseteq \bigcap_{g \notin N} \ker(\rho_g)$*, for some finite dimensional representations* $\rho_g \in \operatorname{rep}(G)$*, such that* $\rho_g(g) \neq 1$*, for every* $g \notin N$*. Then,* G/N *is residually linear.*

Proof. For every $g \in G$, such that $\pi(g)$ is not identity, where π is the canonical projection, we have $g \notin N$. Now, consider a corresponding $\rho_g \in \operatorname{rep}(G)$, then the result holds by the universal property of quotient groups related to ρ_g. □

Theorem 2.36. *Let* G *be a group. The following statements are equivalent.*

(a) *G is residually linear.*

(b) *There exists a sequence*

$$\{e\} = G_1 \subseteq \cdots \subseteq G_i \subseteq \cdots \subseteq G_n = G$$

of subgroups, such that G_{i-1} *is normal in* G_i *and* G_i/G_{i-1} *is residually linear, for every* $1 \leq i \leq n$*,* $n \in \mathbb{N}^*$*.*

To prove the above theorem, let us start first by proving the following Lemma.

Lemma 2.37. *Let G be a group and $N \lhd G$. If N and G/N are residually linear subgroup and quotient group respectively. Then, G is residually linear.*

Proof. N is a residually linear subgroup implies that $N \cap I_G = \{e\}$. G/N being residually linear, we have that $I_G \subseteq N$. Finally, $I_G = \{e\}$, hence as required. □

Now, we prove Theorem 2.36.

Proof. $(a) \Rightarrow (b)$ Assume that G is residually linear, then $\{e\} \subseteq G$ is the desired sequence.
$(b) \Rightarrow (a)$ Let $\{e\} = G_1 \subseteq \cdots \subseteq G_i \subseteq \cdots \subseteq G_n = G$ be a sequence, such that G_{i-1} is normal in G_i and G_i/G_{i-1} is residually linear, for every i. We have $G_2 \simeq G_2/G_1$, so G_2 is residually linear. But G_3/G_2 is residually linear. By Lemma 2.37, G_3 is also residually linear. Hence, by the same procedure progressively for the rest of the sequence, we get that G is residually linear. □

Example 2.38. *Any Noetherian solvable group is residually linear. In fact, as every solvable group has normal series with abelian factor groups that are also finitely generated as we assume the group is Noetherian, then by Theorem 2.36, this group must be residually linear as these factor groups are as well, see Example 2.27 (c).*

Let G be a group. For every $e \neq g \in G$, denote by $\deg_R(g)$ the minimal dimension of the representations $\rho \in \mathrm{rep}(G)$, satisfying $\rho(g) \neq 1$.

Definition 2.39. *Let G be a group. Denote by $\deg_R(G)$, and call residual linearity degree of G over $\Bbbk$, the integer, whenever existed, defined by*

$$\deg_R(G) = \max_{g \in G}\{\deg_R(g)\}.$$

If such integer does not exist, we put $\deg_R(G) = \infty$.

Theorem 2.40. *A group* G *is residually linear if and only if* $\deg_R(G) < \infty$.

Proof. Immediate. □

Remark 2.41. *We make the following observations:*

(a) *Let* G *be a residually linear group and* N *a subgroup of* G. *Denote by* $\deg_R^N(N)$ (*resp.,* $\deg_R^G(N)$), *the residual linearity degree of* N *considered as a group* (*resp., as a subgroup of* G). *Then, over* $\Bbbk$, *we have* $\deg_R^N(N) \leq \deg_R^G(N) \leq \deg_R^G(G)$.

(b) *If* G *admits a faithful representation over a* $\Bbbk-$*vector space of dimension* n, *then, over* $\Bbbk$: $\deg_R(G) \leq n$. *Hence,* $\deg_R(G) \leq \mathrm{rdim}(G)$, *where rdim*($G$) *stands for the representation dimension of* G, *which is the minimal dimension of a finite dimensional faithful representation of* G. *Consequently, if* G *is any group such that* $\mathrm{rdim}(G) < \infty$, *then* G *is residually linear.*

(c) *Let* G *be a group and* $n \in \mathbb{N}^*$. *If* G *does not have any representation on a vector space* V, *with* $\dim(V) = n$. *Then,* $\deg_R(G) > n$.

(d) *Let* G *be a group with* $\deg_R(G) = n$ *and* $N \lhd G$. *Then,* $\deg_R(G/N) \geq n$. *It suffices to remark that* $\deg_R(\overline{g}) \geq \deg_R(g)$, *for every* $g \in G \setminus N$.

(e) *Let* G *be a group and consider the function* $f : G \to \mathbb{N}^*$, $g \mapsto \deg_R(g)$. *Then,* G *is residually linear if and only if* f *is bounded. In this case,* $\deg_R(G)$ *is its upper bound.*

References

[1] M. Artin, *Algebra*, Prentice Hall (1991).

[2] G. Baumslag, "Automorphism groups of residually finite groups", *J. London Math. Soc.*, **38**(1), (1963), 117-118.

[3] A. J. Berrick, "Groups with no nontrivial linear representations", *Bull. Austral. Math. Soc.*, **50**(01), (1994), 1-11.

[4] T. Brocker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, Berlin (1985).

[5] P. de la Harpe and D. McDuff, "Acyclic groups of automorphisms", *Comment. Math. Helv.*, **58**(1), (1983), 48-71.

[6] C. Drutu and M. Sapir, "Non-linear residually finite groups", *J. Algebra*, **284**(1), (2005), 174-178.

[7] E. Formanek and C. Procesi, "The automorphism group of a free group is not linear", *J. Algebra*, **149**(2), (1992), 494-499.

[8] R. Gray and N. Ruškuc, "On residual finiteness of monoids, their Schutzenberger groups and associated actions", *J. Algebra*, **407**, (2014), 21-45 .

[9] K. Gruenberg, "Residual properties of infinite soluble groups", *Proc. Lond. Math. Soc.*, **s3-7**(1), (1957), 29-62.

[10] D. Krammer, "Braid groups are linear", *Ann. of Math.*, **155**(1), (2002), 131-156.

[11] W. Magnus, "Residually finite groups", *Bull. Amer. Math. Soc.*, **75**, (1969), 305-316.

[12] A. Malcev, "On isomorphic matrix representations of infinite groups", *Rec. Math.* [*Mat. Sbornik*] *N.S.*, **8**(50), (1940), 405-422.

[13] P. Menal, "Residual linearity for certain nilpotent groups", *Proc. Amer. Math. Soc.*, **68**, (1978), 27-31.

[14] C. Miller, G. O'Reilly, M. Quick and N. Ruškuc, "On separability finiteness conditions in semigroups", *J. Australian Math. Soc.*, **113**3), (2022), 402-430.

[15] B.H. Neumann, "The isomorphism problem for algebraically closed groups", *Word Probl., Decision Probl. Burnside Probl. Group Theory, Stud. Logic Found. Math.*, **71**, (1973), 553-562.

[16] G. Oman and A. Salminen, "Residually small commutative rings", *J. Commut. Algebra*, **10**(2), (2018), 187-211.

[17] A. Robert, *Introduction to the Representation Theory of Compact and Locally Compact Groups*, London Math. Soc. Lecture Note Series 80, Cambridge Univ. Press (1983).

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